C12 October 2016 (MA)

Q1) $\int f(x)\,dx = \int (3x^2 + x - 4x^{-\frac{1}{2}} + 6x^{-3})\,dx$

$$= x^3 + \frac{x^2}{2} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{6x^{-2}}{-2} + c$$

$$= \boxed{x^3 + \frac{x^2}{2} - 8x^{\frac{1}{2}} - 3x^{-2} + c}$$

Q2a) $7^{2x} = 14$

$\log(7^{2x}) = \log(14)$

$2x \log 7 = \log 14$

$2x = \frac{\log 14}{\log 7}$ $\therefore x = \frac{1}{2} \times \frac{\log 14}{\log 7} \approx \boxed{0.678}$

b) $\log_5(3x+1) = -2$

$5^{-2} = 3x + 1$

$\frac{5^{-2} - 1}{3} = x = \frac{\frac{1}{25} - 1}{3} = \boxed{-\frac{8}{25}}$

Q3 i) $\sqrt{45} - \frac{20}{\sqrt{5}} + \sqrt{6}\cdot\sqrt{30} = \sqrt{45} - \frac{20\sqrt{5}}{5} + \sqrt{6\times 30}$

$= \sqrt{9}\times\sqrt{5} - (\sqrt{5}\times 4) + \sqrt{180}$

$= 3\sqrt{5} - 4\sqrt{5} + \sqrt{5}\times\sqrt{36}$

$= 3\sqrt{5} - 4\sqrt{5} + 6\sqrt{5} = (6+3-4)\sqrt{5}$

$= \boxed{5\sqrt{5}}$

ii) $\frac{17\sqrt{2}}{\sqrt{2}+6} \times \frac{(\sqrt{2}-6)}{(\sqrt{2}-6)}$

$= \frac{17\sqrt{2}(\sqrt{2}-6)}{(\sqrt{2}+6)(\sqrt{2}-6)} = \frac{17(2) - 6(17)\sqrt{2}}{2 - 6\sqrt{2} + 6\sqrt{2} - 36}$

$= \frac{34 - 102\sqrt{2}}{-34} = \boxed{-1 + 3\sqrt{2}}$

Q4 a i) $2x + 1 = 0$

$2x = -1$

$x = -\frac{1}{2}$ // so $f(-\frac{1}{2}) = 6(-\frac{1}{2})^3 - 7(-\frac{1}{2})^2 - 43(-\frac{1}{2}) + 30$

$= \boxed{49}$

ii) $x - 3 = 0$

$x = 3$ // so $f(3) = 6(3)^3 - 7(3)^2 - 43(3) + 30$

$= -30 + 30 = \boxed{0}$

b) $f(3) = 0$ so $(x-3)$ is a factor.

$$\begin{array}{r l}
 & 6x^2 + 11x - 10 \\
x - 3 & \overline{)\,6x^3 - 7x^2 - 43x + 30} \\
 & 6x^3 - 18x^2 \\
 & \overline{0 + 11x^2 - 43x} \\
 & 11x^2 - 33x \\
 & \overline{0 \quad -10x + 30} \\
 & -10x + 30 \\
 & \overline{0 \quad\quad 0}
\end{array}$$

//

$\therefore f(x) = (x-3)(6x^2 + 11x - 10)$

but $6x^2 + 11x - 10 = (2x+5)(3x-2)$ //

so $f(x) = \boxed{(x-3)(2x+5)(3x-2)}$

Q5a) $\left(3 - \frac{ax}{2}\right)^5 \simeq (3)^5 + \binom{5}{1}(3)^4\left(-\frac{ax}{2}\right)^1$

$+ \binom{5}{2}(3)^3\left(-\frac{ax}{2}\right)^2$

$+ \binom{5}{3}(3)^2\left(-\frac{ax}{2}\right)^3$

$\simeq \boxed{243 - \frac{405ax}{2} + \frac{135a^2x^2}{2} - \frac{45}{4}a^3x^3}$

b) coeff. of x is $-\frac{405}{2}a$

coeff. of x^3 is $-\frac{45}{4}a^3$

they are equal so $-\frac{405}{2}a = -\frac{45}{4}a^3$

$\Rightarrow -\frac{405}{2} = -\frac{45}{4}a^2$

$\Rightarrow a^2 = \dfrac{\frac{405}{2}}{\frac{45}{4}} = 18$ //

$\therefore a = \sqrt{18} = \boxed{3\sqrt{2}}$ $(a > 0)$.

Q6 a) $u_2 = \frac{2}{3}(36) = \boxed{24}$

$u_3 = \frac{2}{3}(24) = \boxed{16}$

$u_4 = \frac{2}{3}(16) = \boxed{\frac{32}{3}}$

Geometric series!

$a = 36$
$r = \frac{2}{3}$

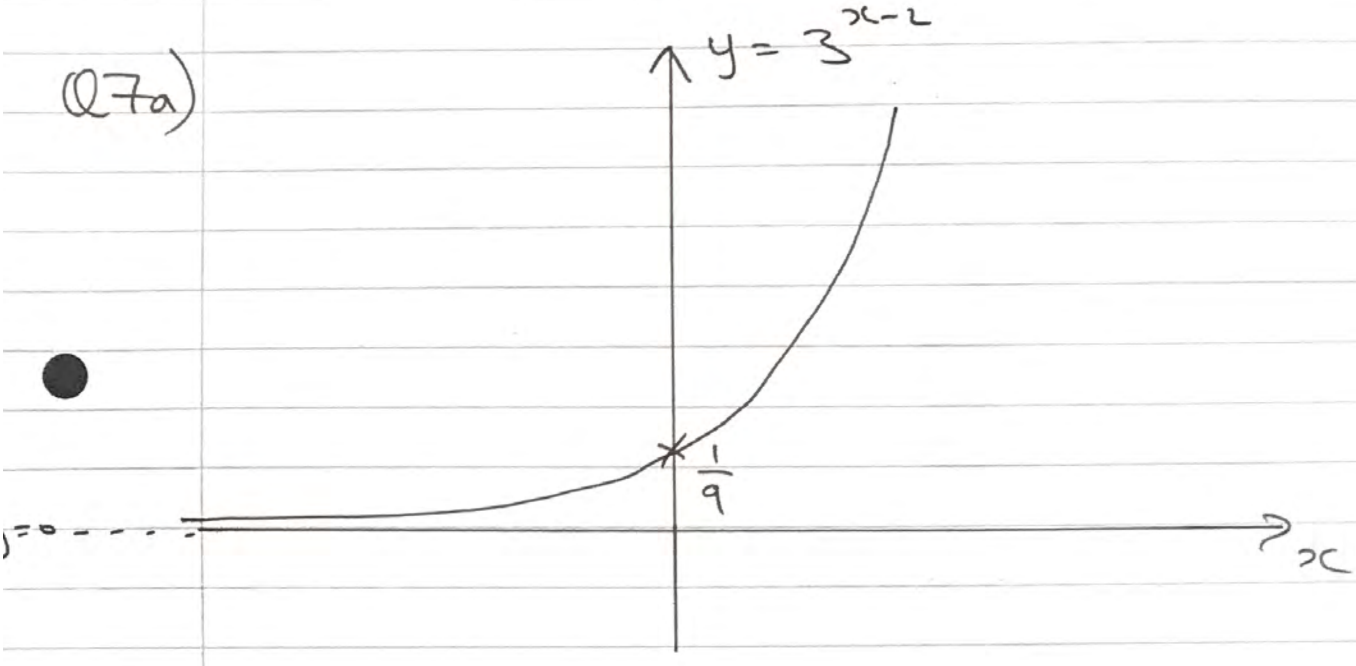
b) $\boxed{\frac{2}{3}}$

c) $u_{11} = ar^{11-1} = 36 \times \left(\frac{2}{3}\right)^{10} = \boxed{0.6243}$

d) $S_6 = \dfrac{a(1-r^6)}{1-r} = \dfrac{36\left(1-\left(\frac{2}{3}\right)^6\right)}{1-\frac{2}{3}} = \boxed{\frac{2660}{27}}$

e) $S_\infty = \frac{a}{1-r} = \frac{36}{1-\frac{2}{3}} = \boxed{108}$

Q7a)

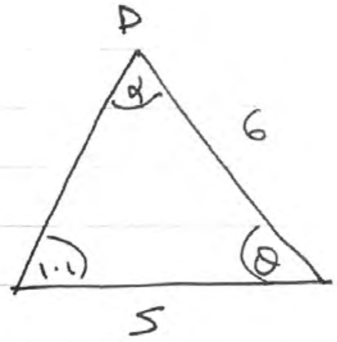


b) $h = \frac{b-a}{n} = \frac{3-0.5}{5} = \frac{2.5}{5} = \frac{1}{2}$

$\therefore$ Area $\simeq \frac{1}{2} \times \frac{1}{2}\left[0.192 + 3 + 2(0.333 + 0.577 + 1 + 1.732)\right]$

$\simeq \boxed{2.62}$

Q8a) sine rule



$$\frac{\sin\alpha}{5} = \frac{\sin 1.1}{6}$$

$$\sin\alpha = \frac{5}{6}\sin 1.1$$

$$\alpha = \sin^{-1}\left(\frac{5}{6}\sin 1.1\right)$$

so θ = angle required $= \pi - 1.1 - \sin^{-1}\left(\frac{5}{6}\sin 1.1\right)$

$$= \boxed{1.20}$$

b) $\angle DBC = \pi - 1.20 = 1.94$

so Area BCD $= \frac{1}{2}(6)^2(1.94) = 34.9$

and Area ABD $= \frac{1}{2}(6)(5)\sin(1.20) = 14.0$

$$34.9 + 14.0 = \boxed{48.9}\ \text{cm}^2$$

Q9a) Sum of first 10 terms $= \frac{10}{2}(2a+9d) = 395$

$$\Rightarrow 5(2a+9d) = 395$$

$$\Rightarrow 10a + 45d = 395 \quad \text{— } (1)$$

b) $S_{18} = \frac{18}{2}(2a+17d) = 927$

$$18a + 153d = 927 \quad \text{— } (2)$$

c) (2) : $a = \frac{927 - 153d}{18}$

→ (1) : $10\left(\frac{927-153d}{18}\right) + 45d = 395$

$$515 - 85d + 45d = 395$$

$$-40d = -120$$

$$\therefore d = \frac{120}{40} = \boxed{3}$$

so $a = \frac{927 - 153(3)}{18} = \boxed{26}$

d) n^{th} term $= a + (n-1)d$

20^{th} term $= 26 + 19(3) = \boxed{83}$

Q10a) $\frac{8\sin x}{\cos x} = -3\cos x$

$\underline{\times \cos x}$: $8\sin x = -3\cos^2 x$

$8\sin x = -3(1-\sin^2 x)$

$8\sin x = -3 + 3\sin^2 x$

$\therefore \quad 3\sin^2 x - 8\sin x - 3 = 0$

b) Same eqn as in (a) but 2θ instead of x.

$\Rightarrow \quad 3\sin^2(2\theta) - 8\sin(2\theta) - 3 = 0$

<u>By Quadratic formula...</u>

$\left.\begin{array}{l} a=3 \\ b=-8 \\ c=-3 \end{array}\right\}$ $[\sin 2\theta = 3]$ or $[\sin 2\theta = -\frac{1}{3}]$

<u>NO SOLUTIONS</u>

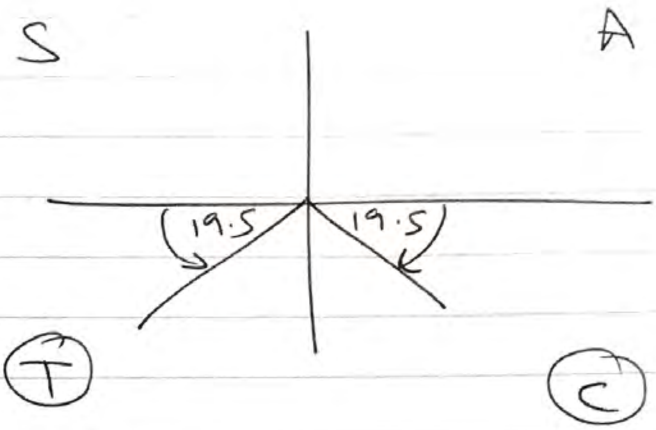
$\sin 2\theta \leq 1$

$\sin 2\theta = -\frac{1}{3}$

$\sin^{-1}(-\frac{1}{3}) = 2\theta = -19.47°$

<u>range</u> : $0 \leq 2\theta < 720°$

$2\theta = 180 + 19.5,$
$360 - 19.5,$
$540 + 19.5,$
$720 - 19.5,$



so $2\theta = 199.47°, 340.53°, 559.47°, 700.53°$

$$\boxed{\theta = 99.7°, 170.3°, 279.7°, 350.3°}$$

Q11a) $5x^2 + 6 = k(13x^2 - 12x)$

$5x^2 - 13kx^2 + 6 = -12kx$

$(5 - 13k)x^2 + (12k)x + 6 = 0$

But this eqn has two distinct real roots,

so $b^2 - 4ac > 0$

$a = 5 - 13k$
$b = 12k$
$c = 6$

$b^2 - 4ac = (12k)^2 - 4(5-13k)(6)$
$= 144k^2 - 24(5) + 24(13k)$

$\therefore b^2 - 4ac = 144k^2 + 312k - 120$

$b^2 - 4ac > 0$ hence $144k^2 + 312k - 120 > 0$

$\div 24$: $6k^2 + 13k - 5 > 0$

b) $6u^2 + 13u - 5 > 0$

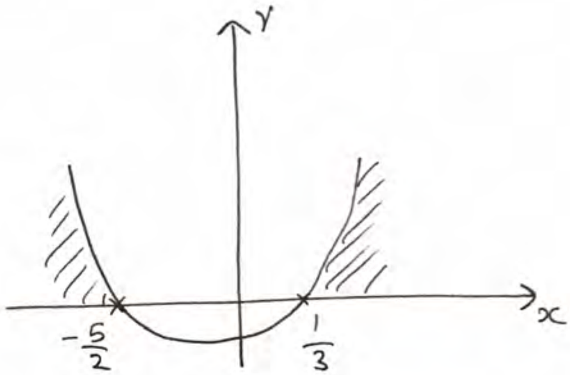
finding critical values... $(6u^2 + 13u - 5 = 0)$ solve

$a = 6$
$b = 13$
$c = -5$

By Quadratic Formula,

$u = \frac{1}{3}$, $u = -\frac{5}{2}$

So $y = 6u^2 + 13u - 5$



$6u^2 + 13u - 5 > 0$ represented by the regions where $y > 0$.

$\therefore$ $\boxed{u > \frac{1}{3} \quad u < -\frac{5}{2}}$

Q12a) $y=0$: $0 = \frac{x^3 - 9x^2 - 81x}{27}$

$\Rightarrow \quad x^3 - 9x^2 - 81x = 0$

$\Rightarrow \quad x(x^2 - 9x - 81) = 0$

$x \neq 0$ (at A & B) so $x^2 - 9x - 81 = 0$

By Quadratic Formula... $x = \frac{9 \pm 9\sqrt{5}}{2}$

$\begin{pmatrix} a = 1 \\ b = -9 \\ c = -81 \end{pmatrix}$ so $\boxed{A\left(\frac{9-9\sqrt{5}}{2}, 0\right) \quad B\left(\frac{9+9\sqrt{5}}{2}, 0\right)}$

b) $y = \frac{1}{27}(x^3 - 9x^2 - 81x)$

at C/D, $\frac{dy}{dx} = 0$ as they are turning points.

$\frac{dy}{dx} = \frac{1}{27}(3x^2 - 18x - 81) = 0$

$3x^2 - 18x - 81 = 0$

$(3x - 27)(x + 3) = 0$

$(0 = 3x - 27) \longrightarrow x = 9$ $\quad (x + 3 = 0) \longrightarrow x = -3$

so C has x coordinate -3.

$$f(-3) = \frac{(-3)^3 - 9(-3)^2 - 81(-3)}{27}$$

$$= 5$$

$\therefore$ $\boxed{C(-3, 5)}$

and D has x coordinate 9

$$f(9) = \frac{(9)^3 - 9(9)^2 - 81(9)}{27}$$

$$= -27$$

$\therefore$ $\boxed{D(9, -27)}$

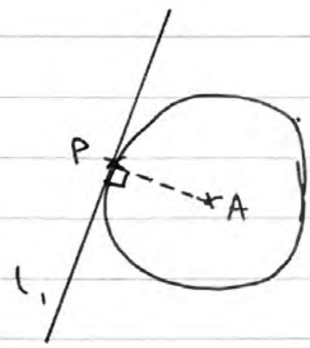
c) D is transformed to the y-axis.
ie 9 units to the left.
so $\boxed{a = 9}$

Q13a) distance AP = $\sqrt{(1-8)^2+(-3+2)^2} = \sqrt{50}$ //

$\therefore r = \sqrt{50}$

so $\boxed{(x-1)^2+(y+3)^2 = 50}$

b) l_1 is perpendicular to AP.



$m_{AP} = \dfrac{-2--3}{8-1} = \dfrac{1}{7}$ //

so $m_{l_1} = -7$ // $\left(-7 \times \frac{1}{7} = -1\right)$

$\therefore y--2 = -7(x-8)$

$y = -7x+56-2$

$\boxed{y = -7x+54}$

c) sub $y = x+6$ into eqn for C.

$(x-1)^2+(x+6+3)^2 = 50$

$(x-1)^2+(x+9)^2 = 50$

$x^2-2x+1+x^2+18x+81 = 50$

$2x^2+16x+82-50 = 0$

$2x^2 + 16x + 32 = 0$

$x^2 + 8x + 16 = 0$

$(x+4)^2 = 0$

$x = -4$

at $x = -4$, $y = (-4) + 6 = 2$

so $\boxed{Q(-4, 2)}$

Q14a) $y = -x^2 + 6x - 8$

$\frac{dy}{dx} = -2x + 6$

at P, $m = -2(5) + 6 = 6 - 10 = -4$

so at normal to P, $m = \frac{1}{4}$ $\left(\frac{1}{4} \times -4 = -1\right)$

$y - -3 = \frac{1}{4}(x - 5)$

$y + 3 = \frac{1}{4}x - \frac{5}{4}$

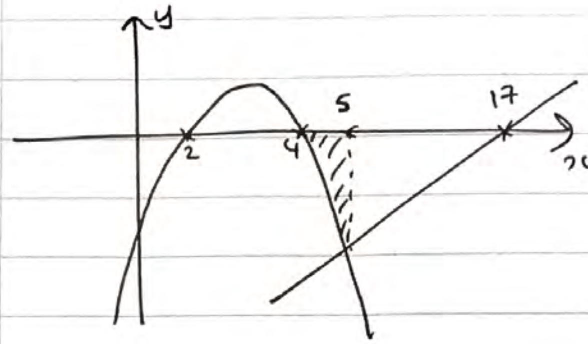
$y = \frac{1}{4}x - \frac{17}{4}$

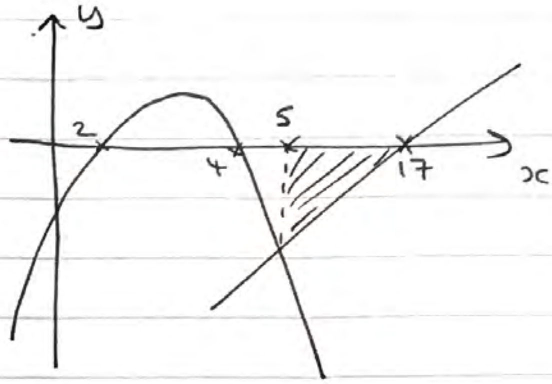
$\underline{\times 4}$: $4y = x - 17$

$\boxed{x - 4y - 17 = 0}$

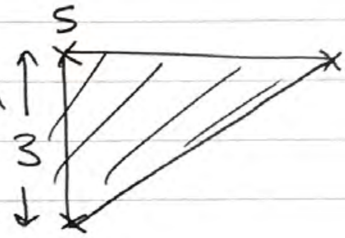
at $y=0$, $\underline{x=4}$, $\underline{x=2}$

b)

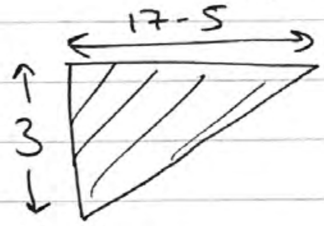
$R =$ 

$+$ 

$$R = \int_4^5 (-x^2+6x-8)\,dx \quad + \quad \text{Area}$$



$$= \left[-\frac{x^3}{3}+3x^2-8x\right]_4^5 \quad +$$



$$= \left[-\frac{20}{3}\right] - \left[-\frac{16}{3}\right] + \frac{1}{2}(3)(12)$$

$$= \left|-\frac{4}{3}\right| + \frac{36}{2} = 18 + \frac{4}{3}$$

$$= \boxed{\frac{58}{3}}$$

Q15a) Surface Area $= \frac{\pi r^2}{2} + \frac{\pi r^2}{2} + \frac{1}{2}(2\pi r h) + h(2r)$

$= \pi r^2 + \pi r h + 2hr$ //

$\therefore\ 200 = \pi r^2 + \pi r h + 2hr$

the Volume equation we need to prove does not include h, so make h the subject here:

$200 = \pi r^2 + h(\pi r + 2r)$

$h = \frac{200 - \pi r^2}{r(\pi + 2)}$ //

and $V = \frac{\pi r^2 h}{2} = \frac{\pi r^2}{2}\left(\frac{200 - \pi r^2}{r(\pi + 2)}\right)$

$\Rightarrow V = \frac{\pi r^{\cancel{2}}(200 - \pi r^2)}{2\cancel{r}(\pi + 2)} = \frac{\pi r(200 - \pi r^2)}{4 + 2\pi}$

just a constant.

b) $V = \left(\frac{\pi}{4+2\pi}\right)\left[200r - \pi r^3\right]$

$$\frac{dV}{dr} = \frac{\pi}{4+2\pi}\left[200 - 3\pi r^2\right] = 0$$

(at max value, $\frac{dV}{dr} = 0$)

$\Rightarrow \quad 200 - 3\pi r^2 = 0$

$\Rightarrow \quad 200 = 3\pi r^2$

$\Rightarrow \quad r^2 = \frac{200}{3\pi}$

$\Rightarrow \quad r = \sqrt{\frac{200}{3\pi}} \quad (= 4.6)$

so $V_{max} = \frac{\pi}{4+2\pi}\left[200\sqrt{\frac{200}{3\pi}} - \pi\left(\sqrt{\frac{200}{3\pi}}\right)^3\right]$

$= \boxed{188}$ cm3

c) $\frac{d^2V}{dr^2} = \frac{\pi}{4+2\pi}\left[-6\pi r\right] < 0$

for all r

hence value of V found is a maximum.